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Consistent superconformal boundary states

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Abstract
We propose a supersymmetric generalization of Cardy's equation for consistent $N = 1$ superconformal boundary states. We solve this equation for the superconformal minimal models $\mathcal{SM}(p/p+2)$ with p odd, and thereby provide a classification of the possible superconformal boundary conditions. In addition to the Neveu–Schwarz (NS) and Ramond boundary states, there are $\widetilde{\mathrm{NS}}$ states. The NS and $\widetilde{\mathrm{NS}}$ boundary states are related by a Z_2 'spin-reversal' transformation. We treat the tricritical Ising model as an example, and in an appendix we discuss the (non-superconformal) case of the Ising model.

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1. Introduction

Two fundamental developments of two-dimensional conformal field theory (CFT) [1, 2] have been the incorporation of supersymmetry [3] and the extension to manifolds with boundary [4]. The concept of a conformal boundary state [5] is of central importance in the formulation of boundary CFT. Hence, in string theory [6, 7], boundary states also figure prominently [8]. (For further references and recent reviews of the boundary state formalism for describing D-branes, see e.g. [9].) The non-supersymmetric (Virasoro algebra) case is well understood [5]: at 'tree' level, conformal invariance implies the constraint

$$\left(L_n - \bar{L}_{-n}\right)|\alpha\rangle = 0 \tag{1.1}$$

on the boundary state $|\alpha\rangle$. This equation has a vector space of solutions which is spanned by the so-called Ishibashi states [10]. For the conformal minimal models, there is an Ishibashi state $|j\rangle\rangle$ corresponding to each chiral primary field $\Phi_j(z)$ (or Virasoro highest-weight representation with highest weight j),

$$|j\rangle\rangle = \sum_N |j\,;N\rangle \otimes U\overline{|j\,;N\rangle} \tag{1.2}$$

where U is an antiunitary operator satisfying $U^\dagger \bar{L}_n U = \bar{L}_n$, and $|j;N\rangle$ is an orthonormal basis of the representation.

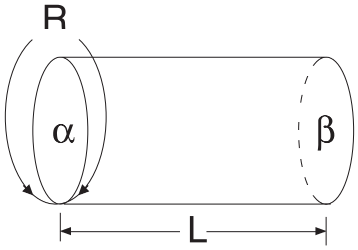

Figure 1. Cylinder of length L and circumference R.

There is a further consistency constraint

$$\operatorname{tr} \mathrm{e}^{-\mathsf{R} H_{\alpha\beta}^{\text{open}}} = \langle\alpha|\mathrm{e}^{-\mathsf{L} H^{\text{closed}}}|\beta\rangle \tag{1.3}$$

which arises for the model on a flat cylinder of length L and circumference R, as represented in figure 1. Here $H_{\alpha\beta}^{\text{open}} = \frac{\pi}{\mathsf{L}}(L_0 - \frac{c}{24})$ is the Hamiltonian in the 'open' channel, with spatial boundary conditions (BCs) denoted by α and β; and $H^{\text{closed}} = \frac{2\pi}{\mathsf{R}}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$ is the Hamiltonian in the 'closed' channel. In the string literature, a similar constraint (with integrations with respect to the corresponding moduli) is known as 'world-sheet duality' or 'open/closed string duality'. The LHS of equation (1.3) can be expressed as

$$\operatorname{tr} \mathrm{e}^{-\mathsf{R} H_{\alpha\beta}^{\text{open}}} = \sum_i N_{\alpha\beta}^i \chi_i(q) \tag{1.4}$$

where the Virasoro characters $\chi_i(q)$ are defined as

$$\chi_i(q) = \operatorname{tr}_i q^{L_0 - \frac{c}{24}} \tag{1.5}$$

and $q = \mathrm{e}^{-\pi\mathsf{R}/\mathsf{L}}$. Under the modular transformation S, the characters transform according to

$$\chi_i(q) = \sum_j S_{ij}\chi_j(\tilde{q}) \tag{1.6}$$

where $\tilde{q} = \mathrm{e}^{-4\pi\mathsf{L}/\mathsf{R}}$. Thus,

$$\operatorname{tr} \mathrm{e}^{-\mathsf{R} H_{\alpha\beta}^{\text{open}}} = \sum_{i,j} N_{\alpha\beta}^i S_{ij}\chi_j(\tilde{q}). \tag{1.7}$$

Expressing the RHS of equation (1.3) in the Ishibashi basis, one obtains

$$\langle\alpha|\mathrm{e}^{-\mathsf{L} H^{\text{closed}}}|\beta\rangle = \sum_j \langle\alpha|j\rangle\!\rangle\langle\!\langle j|\beta\rangle\chi_j(\tilde{q}) \tag{1.8}$$

assuming that each representation j appears once in the spectrum of H^{closed}. Comparing equations (1.7) and (1.8), one arrives at the Cardy equation

$$\sum_i N_{\alpha\beta}^i S_{ij} = \langle\alpha|j\rangle\!\rangle\langle\!\langle j|\beta\rangle. \tag{1.9}$$

Cardy solved this equation for the consistent boundary states

$$|\boldsymbol{k}\rangle = \sum_j \frac{S_{kj}}{\sqrt{S_{0j}}}|j\rangle\!\rangle. \tag{1.10}$$

Moreover, with the help of the Verlinde formula [11], Cardy identified $N_{\boldsymbol{kl}}^i$ as the fusion rule coefficients for $\Phi_k \times \Phi_l \to \Phi_i$. The important result (1.10) provides a classification of the possible conformal BCs for the minimal models, and gives explicit values for the corresponding g-factors [12],

$$g_{\boldsymbol{k}} = \langle\!\langle 0|\boldsymbol{k}\rangle = \frac{S_{k0}}{\sqrt{S_{00}}}. \tag{1.11}$$

Renormalization-group (RG) flows between the various conformal BCs have been investigated in integrable boundary field theories (see e.g. [13–16], and references therein).

The aim of this paper is to generalize the above considerations to the case of $N = 1$ superconformal field theory [3], which encompasses many important models, including superstrings. Some progress on this problem has been made by Apikyan and Sahakyan in [17]. We have been motivated in part by our effort to better understand RG boundary flows in supersymmetric integrable boundary field theories [18, 19].

It is evident that Cardy's results cannot be naively carried over to the supersymmetric case. Indeed, (1.11) would imply that the g-factor of any Ramond boundary state is zero, since modular S matrix elements between Ramond (R) and Neveu–Schwarz (NS) representations generally vanish (see equation (2.12) below).

Although for the Virasoro algebra case the consistent boundary states are in one-to-one correspondence with the irreducible representations, this is no longer true for the superconformal algebra case. Indeed, we find that in the latter case there are more such boundary states. This can be traced to the fact that under S modular transformation, R characters do not transform into NS characters, but rather, into new characters denoted by $\widetilde{\mathrm{NS}}$. The NS and $\widetilde{\mathrm{NS}}$ Cardy states are related by a Z_2 'spin-reversal' transformation, as are the 'fixed +' and 'fixed −' boundary states of the Ising model (IM).

The outline of this paper is as follows. In section 2, we briefly review some necessary results about the $N = 1$ superconformal algebra, its representations, and the modular transformation properties of its characters. In section 3, we formulate a supersymmetric generalization of Cardy's equation, and we find its solutions. We also identify certain coefficients which appear in the super Cardy equations with the fusion rule coefficients of the chiral primary superconformal fields. In section 4 we work out in detail the case of the tricritical Ising model (TIM). This example also serves as a check on our general formalism, since the TIM is also a member of the conformal minimal series. In section 5, we briefly discuss some implications of our results, and mention several possible further generalizations. In an appendix we present an extended discussion of the case of the IM. Although the IM does not have superconformal invariance, it does have NS and R sectors, and it provides valuable insight into how to treat the sectors of superconformal models.

2. Superconformal representation theory

In this section, we first briefly review the superconformal algebra and its representations [3]. We then recall how the characters [20] transform under S modular transformations [21, 22].

The $N = 1$ superconformal algebra is defined by the (anti-) commutation relations

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}c(m^3-m)\delta_{m+n,0} \\
&[L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r} \\
&\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{3}c(r^2-\tfrac{1}{4})\delta_{r+s,0}
\end{aligned} \tag{2.1}$$

where $r, s \in \mathsf{Z}$ for the R sector and $r, s \in \mathsf{Z} + \frac{1}{2}$ for the NS sector. The two modings of G_r are consistent with the Z_2 symmetry ($L_n \to L_n, G_r \to -G_r$) of the algebra. Highest-weight irreducible representations are generated from highest-weight states $|\Delta\rangle$ satisfying

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_n|\Delta\rangle = G_r|\Delta\rangle = 0 \qquad n > 0 \quad r > 0. \tag{2.2}$$

For simplicity, we restrict to the superconformal minimal models that are unitary $\mathcal{SM}(p/p+2)$, for which the central charge c has the values

$$c_p = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \qquad p = 3, 4, \ldots \tag{2.3}$$

and the highest weights Δ are given by

$$\Delta_{(n,m)} = \frac{(n(p+2) - mp)^2 - 4}{8p(p+2)} + \frac{1}{32}(1 - (-1)^{n+m}) \tag{2.4}$$

where $1 \leqslant n \leqslant p-1$ and $1 \leqslant m \leqslant p+1$. The NS representations have $n-m$ even, and the R representations have $n-m$ odd. Following [21][1] we denote by Δ_{NS} and Δ_{R} the following independent sets of NS and R weights, respectively:

$$\begin{aligned} \Delta_{\mathrm{NS}} &= \left\{ \Delta_{(n,m)} | \ 1 \leqslant m \leqslant n \leqslant p-1, n-m \text{ even} \right\} \\ \Delta_{\mathrm{R}} &= \left\{ \Delta_{(n,m)} | \ 1 \leqslant m \leqslant n-1 \text{ for } 1 < n \leqslant (p-1)/2; \ 1 \leqslant m \leqslant n+1 \right. \\ &\qquad \left. \text{for } (p+1)/2 \leqslant n \leqslant p-1, n-m \text{ odd} \right\}. \end{aligned} \tag{2.5}$$

In the R sector, there is a zero mode G_0 which commutes with L_0. Hence, the highest-weight states are generally twofold degenerate, $|\Delta\rangle$ and $G_0|\Delta\rangle$. These states have opposite fermion parity, since G_0 anticommutes with the fermion parity operator $(-1)^{\mathrm{F}}$. Due to the relation $G_0^2 = L_0 - \frac{c}{24}$, if $\Delta = \frac{c}{24}$, then $G_0|\Delta\rangle$ is a null state and decouples, in which case there is a unique highest-weight state.

For p even, there exists an R representation $(n, m) = (\frac{p}{2}, \frac{p+2}{2})$ which has weight $\Delta_{(\frac{p}{2}, \frac{p+2}{2})} = \frac{c_p}{24}$, and so the corresponding highest-weight state is unpaired. For p odd, all the highest-weight states in the R sector are paired.

We define the characters [20–22]

$$\begin{aligned} &\chi_i^{\mathrm{NS}}(q) = \mathrm{tr}_i \, q^{L_0 - \frac{c}{24}} \qquad &&\chi_i^{\widetilde{\mathrm{NS}}}(q) = \mathrm{tr}_i (-1)^{\mathrm{F}} q^{L_0 - \frac{c}{24}} \qquad &&i \in \Delta_{\mathrm{NS}} \\ &\chi_i^{\mathrm{R}}(q) = \mathrm{tr}_i \, q^{L_0 - \frac{c}{24}} \qquad &&\chi_i^{\tilde{\mathrm{R}}} = \mathrm{tr}_i (-1)^{\mathrm{F}} q^{L_0 - \frac{c}{24}} \qquad &&i \in \Delta_{\mathrm{R}}. \end{aligned} \tag{2.6}$$

From the above remarks, it follows that for p odd, $\chi_i^{\tilde{\mathrm{R}}} = 0$ for all representations i; and for p even,

$$\chi_i^{\tilde{\mathrm{R}}} = \pm \delta_{i, \Delta_{(\frac{p}{2}, \frac{p+2}{2})}}. \tag{2.7}$$

The characters transform under the S modular transformation according to [21, 22]

$$\begin{aligned} \chi_i^{\mathrm{NS}}(q) &= \sum_{j \in \Delta_{\mathrm{NS}}} S_{ij}^{[\mathrm{NS},\mathrm{NS}]} \chi_j^{\mathrm{NS}}(\tilde{q}) \\ \chi_i^{\widetilde{\mathrm{NS}}}(q) &= \sum_{j \in \Delta_{\mathrm{R}}} S_{ij}^{[\widetilde{\mathrm{NS}},\mathrm{R}]} \sqrt{2} \chi_j^{\mathrm{R}}(\tilde{q}) \\ \sqrt{2} \chi_i^{\mathrm{R}}(q) &= \sum_{j \in \Delta_{\mathrm{NS}}} S_{ij}^{[\mathrm{R},\widetilde{\mathrm{NS}}]} \chi_j^{\widetilde{\mathrm{NS}}}(\tilde{q}) \end{aligned} \tag{2.8}$$

where $\tilde{q} = \mathrm{e}^{-4\pi \mathsf{L}/\mathsf{R}}$. As already mentioned in the introduction, the characters $\chi_i^{\widetilde{\mathrm{NS}}}$ appear when the characters χ_i^{R} undergo a modular transformation. For the superconformal minimal models $\mathcal{SM}(p/p+2)$ with p odd, the modular S matrices are given by [21]

$$S_{(n,m),(n',m')}^{[\mathrm{NS},\mathrm{NS}]} = \frac{4}{\sqrt{p(p+2)}} \sin \frac{\pi n n'}{p} \sin \frac{\pi m m'}{p+2} \tag{2.9}$$

$$S_{(n,m),(n',m')}^{[\widetilde{\mathrm{NS}},\mathrm{R}]} = \frac{4}{\sqrt{p(p+2)}} (-1)^{(n-m)/2} \sin \frac{\pi n n'}{p} \sin \frac{\pi m m'}{p+2} \tag{2.10}$$

$$S_{(n,m),(n',m')}^{[\mathrm{R},\widetilde{\mathrm{NS}}]} = \frac{4}{\sqrt{p(p+2)}} (-1)^{(n'-m')/2} \sin \frac{\pi n n'}{p} \sin \frac{\pi m m'}{p+2}. \tag{2.11}$$

[1] We generally follow the conventions of Matsuo and Yahikozawa [21], with the main exception that our characters (2.6) have an extra factor $q^{-\frac{c}{24}}$.

These matrices can be arranged into the matrix S

$$S = \begin{pmatrix} S^{[\mathrm{NS,NS}]} & 0 & 0 \\ 0 & 0 & S^{[\widetilde{\mathrm{NS}},\mathrm{R}]} \\ 0 & S^{[\mathrm{R},\widetilde{\mathrm{NS}}]} & 0 \end{pmatrix} \tag{2.12}$$

which is real, symmetric and satisfies $S^2 = \mathbb{I}$. We do not quote the corresponding expressions for the case p even, which are somewhat more complicated due to the special representation $(\frac{p}{2}, \frac{p+2}{2})$.

3. Consistent boundary states

The full operator algebra of the NS and R superconformal primary fields is nonlocal. We consider here the so-called spin model [3] which has a local operator algebra. It is obtained by projecting on even fermion parity $(-1)^{\mathrm{F}} = 1$ in the NS sector, and either even or odd fermion parity $(-1)^{\mathrm{F}} = \pm 1$ in the R sector[2]. For definiteness, we treat only the case with even fermion parity also in the R sector. Also, for simplicity, we restrict ourselves to models all of whose representations satisfy $\Delta \neq \frac{c}{24}$; that is, we consider only the superconformal minimal models $\mathcal{SM}(p/p+2)$ with p odd. Moreover, we again assume that the bulk theory is diagonal, with each representation appearing once.

Our goal is to construct for such spin models the complete set of consistent superconformal boundary states $|\alpha\rangle$, by solving the various constraints which they must obey. The restriction to even fermion parity implies the constraint

$$(-1)^{\mathrm{F}}|\alpha\rangle = |\alpha\rangle \tag{3.1}$$

where here F is the total fermion number of right and left movers. Superconformal invariance implies the constraints [10, 17]

$$\left(L_n - \bar{L}_{-n}\right)|\alpha\rangle = 0 \qquad \left(G_r + \mathrm{i}\gamma \bar{G}_{-r}\right)|\alpha\rangle = 0 \tag{3.2}$$

where γ is either +1 or -1. Finally, we impose the further constraint

$$\mathrm{tr}_{\mathrm{NS}}\, \tfrac{1}{2}(1 + (-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} + \mathrm{tr}_{\mathrm{R}}\, \tfrac{1}{2}(1 + (-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} = \langle\alpha|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|\beta\rangle \tag{3.3}$$

for a spin model on the cylinder in figure 1. The projectors $\frac{1}{2}(1 + (-1)^{\mathrm{F}})$ project onto states of even fermion parity in the open channel. The Hamiltonians in the open and closed channels are (as in the non-supersymmetric case which was reviewed in the introduction) given by $H^{\mathrm{open}}_{\alpha\beta} = \frac{\pi}{\mathsf{L}}(L_0 - \frac{c}{24})$ and $H^{\mathrm{closed}} = \frac{2\pi}{\mathsf{R}}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$, respectively. This constraint is similar to the IM constraint (A.32), except without the term involving the projector $\frac{1}{2}(1 - (-1)^{\mathrm{F}})$ in the NS sector.

We first consider the open channel. We define the coefficients $n^i_{\alpha\beta}$ etc by

$$\begin{aligned}
\mathrm{tr}_{\mathrm{NS}}\, \mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} &= \sum_{i\in\Delta_{\mathrm{NS}}} n^i_{\alpha\beta}\chi^{\mathrm{NS}}_i(q) \\
\mathrm{tr}_{\mathrm{NS}}(-1)^{\mathrm{F}}\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} &= \sum_{i\in\Delta_{\mathrm{NS}}} \tilde{n}^i_{\alpha\beta}\chi^{\widetilde{\mathrm{NS}}}_i(q) \\
\mathrm{tr}_{\mathrm{R}}\, \mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} &= \sum_{i\in\Delta_{\mathrm{R}}} m^i_{\alpha\beta}\chi^{\mathrm{R}}_i(q) \\
\mathrm{tr}_{\mathrm{R}}(-1)^{\mathrm{F}}\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}_{\alpha\beta}} &= \sum_{i\in\Delta_{\mathrm{R}}} \tilde{m}^i_{\alpha\beta}\chi^{\tilde{\mathrm{R}}}_i = 0
\end{aligned} \tag{3.4}$$

[2] An analysis of consistent boundary states for the so-called fermionic model, which is obtained by keeping only the NS sector, is given in [23].

where $q = \mathrm{e}^{-\pi \mathsf{R}/\mathsf{L}}$, and the various characters are defined in (2.6). In the last line, we have made use of our restriction to p odd, together with the result (2.7). It follows that

$$\begin{aligned}
\text{LHS of equation (3.3)} &= \tfrac{1}{2}\sum_{i\in\Delta_{\mathrm{NS}}}\left(n^i_{\alpha\beta}\chi_i^{\mathrm{NS}}(q) + \tilde{n}^i_{\alpha\beta}\chi_i^{\widetilde{\mathrm{NS}}}(q)\right) + \tfrac{1}{2}\sum_{i\in\Delta_{\mathrm{R}}} m^i_{\alpha\beta}\chi_i^{\mathrm{R}}(q) \\
&= \tfrac{1}{2}\sum_{i\in\Delta_{\mathrm{NS}}}\left(\sum_{j\in\Delta_{\mathrm{NS}}} n^i_{\alpha\beta} S_{ij}^{[\mathrm{NS},\mathrm{NS}]}\chi_j^{\mathrm{NS}}(\tilde{q}) + \sum_{j\in\Delta_{\mathrm{R}}}\tilde{n}^i_{\alpha\beta} S_{ij}^{[\widetilde{\mathrm{NS}},\mathrm{R}]}\sqrt{2}\chi_j^{\mathrm{R}}(\tilde{q})\right) \\
&\quad + \frac{1}{2}\sum_{i\in\Delta_{\mathrm{R}}}\sum_{j\in\Delta_{\mathrm{NS}}} m^i_{\alpha\beta} S_{ij}^{[\mathrm{R},\widetilde{\mathrm{NS}}]}\frac{1}{\sqrt{2}}\chi_j^{\widetilde{\mathrm{NS}}}(\tilde{q})
\end{aligned} \tag{3.5}$$

where $\tilde{q} = \mathrm{e}^{-4\pi \mathsf{L}/\mathsf{R}}$. In passing to the second equality, we have made use of the modular transformation properties (2.8) of the characters.

Turning now to the closed channel, we recall [10,17] that, corresponding to each irreducible representation j of the superconformal algebra, one can construct a pair of Ishibashi states $|j_\pm\rangle\rangle$ satisfying

$$\begin{aligned}
\left(L_n - \bar{L}_{-n}\right)|j_\pm\rangle\rangle &= 0 \\
\left(G_r \pm \mathrm{i}\bar{G}_{-r}\right)|j_\pm\rangle\rangle &= 0.
\end{aligned} \tag{3.6}$$

From the explicit expressions for the Ishibashi states, it is easy to see that the states in the NS sector have even fermion parity

$$(-1)^{\mathrm{F}}|j_\pm^{\mathrm{NS}}\rangle\rangle = |j_\pm^{\mathrm{NS}}\rangle\rangle \tag{3.7}$$

where (as in equation (3.1)) F is the total fermion number of right and left movers. For the R sector, the computation of fermion parity is more subtle due to the presence of zero modes [17]. We assume that, in analogy with the IM result (A.25),

$$(-1)^{\mathrm{F}}|j_\pm^{\mathrm{R}}\rangle\rangle = \pm|j_\pm^{\mathrm{R}}\rangle\rangle. \tag{3.8}$$

We propose that the set of Ishibashi states $\{|j_\pm^{\mathrm{NS}}\rangle\rangle, |j_+^{\mathrm{R}}\rangle\rangle\}$ constitutes a basis for the boundary states. That is,

$$|\alpha\rangle = \sum_{j\in\Delta_{\mathrm{NS}}}\left(|j_+^{\mathrm{NS}}\rangle\rangle\langle\langle j_+^{\mathrm{NS}}|\alpha\rangle + |j_-^{\mathrm{NS}}\rangle\rangle\langle\langle j_-^{\mathrm{NS}}|\alpha\rangle\right) + \sum_{j\in\Delta_{\mathrm{R}}}|j_+^{\mathrm{R}}\rangle\rangle\langle\langle j_+^{\mathrm{R}}|\alpha\rangle. \tag{3.9}$$

Indeed, equations (3.7) and (3.8) imply that the constraint (3.1) is already satisfied. For a given value of γ, the constraints (3.2) can be satisfied by keeping in the expansion (3.9) only the terms involving $|j_\gamma\rangle\rangle$, i.e. setting $\langle\langle j_{-\gamma}|\alpha\rangle = 0$. Moreover, the number of basis vectors (twice the number of NS representations plus the number of R representations) is the same as the dimension of the vector space on which the full modular S matrix (2.12) acts, which we expect is the number of consistent boundary states. The expansion (3.9) is also motivated by the corresponding result (A.34) for the IM.

In this basis, we have

$$\begin{aligned}
\text{RHS of equation (3.3)} &= \sum_{j\in\Delta_{\mathrm{NS}}}\left(\langle\alpha|j_+^{\mathrm{NS}}\rangle\rangle\langle\langle j_+^{\mathrm{NS}}|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|j_+^{\mathrm{NS}}\rangle\rangle\langle\langle j_+^{\mathrm{NS}}|\beta\rangle\right. \\
&\quad + \langle\alpha|j_+^{\mathrm{NS}}\rangle\rangle\langle\langle j_+^{\mathrm{NS}}|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|j_-^{\mathrm{NS}}\rangle\rangle\langle\langle j_-^{\mathrm{NS}}|\beta\rangle \\
&\quad + \langle\alpha|j_-^{\mathrm{NS}}\rangle\rangle\langle\langle j_-^{\mathrm{NS}}|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|j_+^{\mathrm{NS}}\rangle\rangle\langle\langle j_+^{\mathrm{NS}}|\beta\rangle \\
&\quad \left. + \langle\alpha|j_-^{\mathrm{NS}}\rangle\rangle\langle\langle j_-^{\mathrm{NS}}|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|j_-^{\mathrm{NS}}\rangle\rangle\langle\langle j_-^{\mathrm{NS}}|\beta\rangle\right) \\
&\quad + \sum_{j\in\Delta_{\mathrm{R}}}\langle\alpha|j_+^{\mathrm{R}}\rangle\rangle\langle\langle j_+^{\mathrm{R}}|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|j_+^{\mathrm{R}}\rangle\rangle\langle\langle j_+^{\mathrm{R}}|\beta\rangle
\end{aligned}$$

$$= \sum_{j\in\Delta_{\rm NS}} \Big[\big(\langle\alpha|j_+^{\rm NS}\rangle\!\rangle\langle\!\langle j_+^{\rm NS}|\beta\rangle + \langle\alpha|j_-^{\rm NS}\rangle\!\rangle\langle\!\langle j_-^{\rm NS}|\beta\rangle\big)\chi_j^{\rm NS}(\tilde q)$$
$$+\big(\langle\alpha|j_-^{\rm NS}\rangle\!\rangle\langle\!\langle j_+^{\rm NS}|\beta\rangle + \langle\alpha|j_+^{\rm NS}\rangle\!\rangle\langle\!\langle j_-^{\rm NS}|\beta\rangle\big)\chi_j^{\widetilde{\rm NS}}(\tilde q)\Big]$$
$$+ \sum_{j\in\Delta_{\rm R}} \langle\alpha|j_+^{\rm R}\rangle\!\rangle\langle\!\langle j_+^{\rm R}|\beta\rangle\chi_j^{\rm R}(\tilde q). \tag{3.10}$$

In passing to the second equality, we have used the relations

$$\begin{aligned}
\langle\!\langle j_\pm^{\rm NS}|{\rm e}^{-{\rm L}H^{\rm closed}}|j_\pm^{\rm NS}\rangle\!\rangle &= \chi_j^{\rm NS}(\tilde q)\\
\langle\!\langle j_\mp^{\rm NS}|{\rm e}^{-{\rm L}H^{\rm closed}}|j_\pm^{\rm NS}\rangle\!\rangle &= \chi_j^{\widetilde{\rm NS}}(\tilde q)\\
\langle\!\langle j_\pm^{\rm R}|{\rm e}^{-{\rm L}H^{\rm closed}}|j_\pm^{\rm R}\rangle\!\rangle &= \chi_j^{\rm R}(\tilde q)\\
\langle\!\langle j_\mp^{\rm R}|{\rm e}^{-{\rm L}H^{\rm closed}}|j_\pm^{\rm R}\rangle\!\rangle &= \chi_j^{\tilde{\rm R}}(\tilde q) = 0
\end{aligned} \tag{3.11}$$

which are analogous to the results (A.26) for the IM.

Comparing equations (3.5) and (3.10), we arrive at the 'super' Cardy equations (cf (1.9))

$$\begin{aligned}
\tfrac{1}{2}\sum_{i\in\Delta_{\rm NS}} n^i_{\alpha\beta} S_{ij}^{[{\rm NS},{\rm NS}]} &= \langle\alpha|j_+^{\rm NS}\rangle\!\rangle\langle\!\langle j_+^{\rm NS}|\beta\rangle + \langle\alpha|j_-^{\rm NS}\rangle\!\rangle\langle\!\langle j_-^{\rm NS}|\beta\rangle\\
\frac{1}{\sqrt{2}}\sum_{i\in\Delta_{\rm NS}} \tilde n^i_{\alpha\beta} S_{ij}^{[\widetilde{\rm NS},{\rm R}]} &= \langle\alpha|j_+^{\rm R}\rangle\!\rangle\langle\!\langle j_+^{\rm R}|\beta\rangle\\
\frac{1}{2\sqrt{2}}\sum_{i\in\Delta_{\rm R}} m^i_{\alpha\beta} S_{ij}^{[{\rm R},\widetilde{\rm NS}]} &= \langle\alpha|j_+^{\rm NS}\rangle\!\rangle\langle\!\langle j_-^{\rm NS}|\beta\rangle + \langle\alpha|j_-^{\rm NS}\rangle\!\rangle\langle\!\langle j_+^{\rm NS}|\beta\rangle.
\end{aligned} \tag{3.12}$$

We now proceed to solve these equations, together with the constraints (3.2), for the consistent superconformal boundary states. Defining the state $|\mathbf{0}^{\rm NS}\rangle$ as the solution with $n^i_{\mathbf{0}^{\rm NS}\mathbf{0}^{\rm NS}} = \tilde n^i_{\mathbf{0}^{\rm NS}\mathbf{0}^{\rm NS}} = \delta^i_0$, $m^i_{\mathbf{0}^{\rm NS}\mathbf{0}^{\rm NS}} = 0$, we obtain

$$|\mathbf{0}^{\rm NS}\rangle = \frac{1}{\sqrt{2}}\sum_{j\in\Delta_{\rm NS}}\sqrt{S_{0j}^{[{\rm NS},{\rm NS}]}}|j_+^{\rm NS}\rangle\!\rangle + \frac{1}{\sqrt[4]{2}}\sum_{j\in\Delta_{\rm R}}\sqrt{S_{0j}^{[\widetilde{\rm NS},{\rm R}]}}|j_+^{\rm R}\rangle\!\rangle. \tag{3.13}$$

We then define the states $|\boldsymbol{k}^{\rm NS}\rangle$ and $|\boldsymbol{k}^{\widetilde{\rm NS}}\rangle$ with $k\in\Delta_{\rm NS}$ by

$$\begin{aligned}
n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm NS}} = \tilde n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm NS}} &= \delta^i_k \qquad m^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm NS}} = 0\\
n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\widetilde{\rm NS}}} = -\tilde n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\widetilde{\rm NS}}} &= \delta^i_k \qquad m^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\widetilde{\rm NS}}} = 0
\end{aligned} \tag{3.14}$$

respectively, and we obtain

$$|\boldsymbol{k}^{\rm NS}\rangle = \frac{1}{\sqrt{2}}\sum_{j\in\Delta_{\rm NS}}\frac{S_{kj}^{[{\rm NS},{\rm NS}]}}{\sqrt{S_{0j}^{[{\rm NS},{\rm NS}]}}}|j_+^{\rm NS}\rangle\!\rangle + \frac{1}{\sqrt[4]{2}}\sum_{j\in\Delta_{\rm R}}\frac{S_{kj}^{[\widetilde{\rm NS},{\rm R}]}}{\sqrt{S_{0j}^{[\widetilde{\rm NS},{\rm R}]}}}|j_+^{\rm R}\rangle\!\rangle \tag{3.15}$$

$$|\boldsymbol{k}^{\widetilde{\rm NS}}\rangle = \frac{1}{\sqrt{2}}\sum_{j\in\Delta_{\rm NS}}\frac{S_{kj}^{[{\rm NS},{\rm NS}]}}{\sqrt{S_{0j}^{[{\rm NS},{\rm NS}]}}}|j_+^{\rm NS}\rangle\!\rangle - \frac{1}{\sqrt[4]{2}}\sum_{j\in\Delta_{\rm R}}\frac{S_{kj}^{[\widetilde{\rm NS},{\rm R}]}}{\sqrt{S_{0j}^{[\widetilde{\rm NS},{\rm R}]}}}|j_+^{\rm R}\rangle\!\rangle. \tag{3.16}$$

Finally, we define the states $|\boldsymbol{k}^{\rm R}\rangle$ with $k\in\Delta_{\rm R}$ by

$$n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm R}} = \tilde n^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm R}} = 0 \qquad m^i_{\mathbf{0}^{\rm NS}\boldsymbol{k}^{\rm R}} = 2\delta^i_k \tag{3.17}$$

and we obtain

$$|\boldsymbol{k}^{\rm R}\rangle = \sum_{j\in\Delta_{\rm NS}}\frac{S_{kj}^{[{\rm R},\widetilde{\rm NS}]}}{\sqrt{S_{0j}^{[{\rm NS},{\rm NS}]}}}|j_-^{\rm NS}\rangle\!\rangle. \tag{3.18}$$

We shall refer to the states (3.15), (3.16) and (3.18) as the NS, $\widetilde{\mathrm{NS}}$ and R Cardy states, respectively. These states manifestly satisfy the constraints (3.2), with the R states and the NS, $\widetilde{\mathrm{NS}}$ states having opposite signs of γ. The NS and $\widetilde{\mathrm{NS}}$ states differ by the Z_2 'spin-reversal' transformation $|j^{\mathrm{NS}}\rangle\!\rangle \to |j^{\mathrm{NS}}\rangle\!\rangle, \quad |j^{\mathrm{R}}\rangle\!\rangle \to -|j^{\mathrm{R}}\rangle\!\rangle$, just like the 'fixed +' and 'fixed −' boundary states of the IM (A.36).

The equations (3.12) and their solutions (3.15), (3.16) and (3.18) are the main results of this paper[3]. These solutions provide a classification of the possible superconformal BCs for the superconformal minimal models $\mathcal{SM}(p/p+2)$ with p odd.

The g-factor [12] of a boundary state $|\alpha\rangle$ is given by

$$g_\alpha = \left(\langle\!\langle 0_+^{\mathrm{NS}}| + \langle\!\langle 0_-^{\mathrm{NS}}|\right)|\alpha\rangle. \tag{3.19}$$

Hence, the g-factors of the Cardy states are

$$g_{k^{\mathrm{NS}}} = g_{k^{\widetilde{\mathrm{NS}}}} = \frac{1}{\sqrt{2}} \frac{S_{k0}^{[\mathrm{NS,NS}]}}{\sqrt{S_{00}^{[\mathrm{NS,NS}]}}} \tag{3.20}$$

$$g_{k^{\mathrm{R}}} = \frac{S_{k0}^{[\mathrm{R},\widetilde{\mathrm{NS}}]}}{\sqrt{S_{00}^{[\mathrm{NS,NS}]}}}. \tag{3.21}$$

We see from (3.20) that, for an NS state, the naive use of the modular S matrix (2.9) in the original Cardy result (1.11) would give a g-factor which is a factor $\sqrt{2}$ too big. Moreover, the g-factor (3.21) of an R state does *not* generally vanish.

As in the non-supersymmetric case, the various coefficients $n_{\alpha\beta}^{i}$ etc in equation (3.4) can now be expressed in terms of modular S matrices and be related to fusion rule coefficients. Indeed, by substituting the expression (3.15) for two NS Cardy states $|k^{\mathrm{NS}}\rangle$ and $|l^{\mathrm{NS}}\rangle$ back into the super Cardy formula (3.12), we obtain

$$\begin{aligned}
n_{k^{\mathrm{NS}}l^{\mathrm{NS}}}^{i} &= \sum_{j\in\Delta_{\mathrm{NS}}} \frac{S_{kj}^{[\mathrm{NS,NS}]} S_{lj}^{[\mathrm{NS,NS}]} \left(S^{[\mathrm{NS,NS}]}\right)_{ji}^{-1}}{S_{0j}^{[\mathrm{NS,NS}]}} \\
\tilde{n}_{k^{\mathrm{NS}}l^{\mathrm{NS}}}^{i} &= \sum_{j\in\Delta_{\mathrm{R}}} \frac{S_{kj}^{[\widetilde{\mathrm{NS}},\mathrm{R}]} S_{lj}^{[\widetilde{\mathrm{NS}},\mathrm{R}]} \left(S^{[\widetilde{\mathrm{NS}},\mathrm{R}]}\right)_{ji}^{-1}}{S_{0j}^{[\widetilde{\mathrm{NS}},\mathrm{R}]}} \\
m_{k^{\mathrm{NS}}l^{\mathrm{NS}}}^{i} &= 0.
\end{aligned} \tag{3.22}$$

From the work [24] (see also [25]) on a generalized Verlinde formula, we can identify $n_{k^{\mathrm{NS}}l^{\mathrm{NS}}}^{i}$ as the fusion rule coefficient for $\Phi_k^{\mathrm{NS}} \times \Phi_l^{\mathrm{NS}} \to \Phi_i^{\mathrm{NS}}$. Similarly, for two R Cardy states (3.18), we obtain

$$\begin{aligned}
n_{k^{\mathrm{R}}l^{\mathrm{R}}}^{i} &= 2\sum_{j\in\Delta_{\mathrm{NS}}} \frac{S_{kj}^{[\mathrm{R},\widetilde{\mathrm{NS}}]} S_{lj}^{[\mathrm{R},\widetilde{\mathrm{NS}}]} \left(S^{[\mathrm{NS,NS}]}\right)_{ji}^{-1}}{S_{0j}^{[\mathrm{NS,NS}]}} \\
\tilde{n}_{k^{\mathrm{R}}l^{\mathrm{R}}}^{i} &= m_{k^{\mathrm{R}}l^{\mathrm{R}}}^{i} = 0
\end{aligned} \tag{3.23}$$

and we identify $n_{k^{\mathrm{R}}l^{\mathrm{R}}}^{i}$ as the fusion rule coefficient for $\Phi_k^{\mathrm{R}} \times \Phi_l^{\mathrm{R}} \to \Phi_i^{\mathrm{NS}}$. Finally, for one NS state and one R state, we obtain

$$\begin{aligned}
m_{k^{\mathrm{NS}}l^{\mathrm{R}}}^{i} &= 2\sum_{j\in\Delta_{\mathrm{NS}}} \frac{S_{kj}^{[\mathrm{NS,NS}]} S_{lj}^{[\mathrm{R},\widetilde{\mathrm{NS}}]} \left(S^{[\mathrm{R},\widetilde{\mathrm{NS}}]}\right)_{ji}^{-1}}{S_{0j}^{[\mathrm{NS,NS}]}} \\
n_{k^{\mathrm{NS}}l^{\mathrm{R}}}^{i} &= \tilde{n}_{k^{\mathrm{NS}}l^{\mathrm{R}}}^{i} = 0
\end{aligned} \tag{3.24}$$

[3] In [17] a different set of equations is proposed, which gives the NS states (3.15), but not the $\widetilde{\mathrm{NS}}$ and R states (3.16), (3.18).

Table 1. Kac table for $\mathcal{SM}(3/5)$.

$\frac{7}{16}$	$\frac{1}{10}$	$\frac{3}{80}$	0
0	$\frac{3}{80}$	$\frac{1}{10}$	$\frac{7}{16}$

and we identify $m^{i}_{k^{\mathrm{NS}}l^{\mathrm{R}}}$ as the fusion rule coefficient for $\Phi^{\mathrm{NS}}_{k} \times \Phi^{\mathrm{R}}_{l} \to \Phi^{\mathrm{R}}_{i}$. The results for the coefficients involving $\widetilde{\mathrm{NS}}$ states (3.16) are very similar to those for the corresponding NS states.

4. Tricritical Ising model

As an example of the general formalism presented in the previous section, we now work out in detail the first nontrivial case: namely, the superconformal minimal model $\mathcal{SM}(3/5)$ ($p = 3$), which has been identified [3] as the TIM. This model is equivalent to the conformal minimal model $\mathcal{M}(4/5)$, for which the Cardy states are already known [5, 14]. Hence, this example also serves as a valuable check on our general formalism.

The Kac table for $\mathcal{SM}(3/5)$, which is obtained using equation (2.4), is given in table 1. The modular S matrix is (2.9)–(2.12)

$$S = \begin{pmatrix} 2a & 2b & 0 & 0 & 0 & 0 \\ 2b & -2a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}c & \sqrt{2}d \\ 0 & 0 & 0 & 0 & -\sqrt{2}d & \sqrt{2}c \\ 0 & 0 & \sqrt{2}c & -\sqrt{2}d & 0 & 0 \\ 0 & 0 & \sqrt{2}d & \sqrt{2}c & 0 & 0 \end{pmatrix} \tag{4.1}$$

where the rows and columns are labelled by the highest weights $(0^{\mathrm{NS}}, \frac{1}{10}^{\mathrm{NS}}, 0^{\widetilde{\mathrm{NS}}}, \frac{1}{10}^{\widetilde{\mathrm{NS}}}, \frac{7}{16}^{\mathrm{R}}, \frac{3}{80}^{\mathrm{R}})$, and where

$$\begin{aligned} a &= \tfrac{1}{2}\sqrt{\tfrac{1}{10}(5-\sqrt{5})} \qquad & b &= \tfrac{1}{2}\sqrt{\tfrac{1}{10}(5+\sqrt{5})} \\ c &= \tfrac{1}{2}\sqrt{\tfrac{1}{5}(5-\sqrt{5})} \qquad & d &= \tfrac{1}{2}\sqrt{\tfrac{1}{5}(5+\sqrt{5})}. \end{aligned} \tag{4.2}$$

According to our results (3.15), (3.16), (3.18), there are six Cardy states, given by

$$\begin{aligned} |\mathbf{0}^{\mathrm{NS}}\rangle &= \sqrt{a}|0^{\mathrm{NS}}_{+}\rangle\!\rangle + \sqrt{b}|\tfrac{1}{10}{}^{\mathrm{NS}}_{+}\rangle\!\rangle + \sqrt{c}|\tfrac{7}{16}{}^{\mathrm{R}}_{+}\rangle\!\rangle + \sqrt{d}|\tfrac{3}{80}{}^{\mathrm{R}}_{+}\rangle\!\rangle \\ |\mathbf{0}^{\widetilde{\mathrm{NS}}}\rangle &= \sqrt{a}|0^{\mathrm{NS}}_{+}\rangle\!\rangle + \sqrt{b}|\tfrac{1}{10}{}^{\mathrm{NS}}_{+}\rangle\!\rangle - \sqrt{c}|\tfrac{7}{16}{}^{\mathrm{R}}_{+}\rangle\!\rangle - \sqrt{d}|\tfrac{3}{80}{}^{\mathrm{R}}_{+}\rangle\!\rangle \\ |\tfrac{\mathbf{1}}{\mathbf{10}}{}^{\mathrm{NS}}\rangle &= \frac{b}{\sqrt{a}}|0^{\mathrm{NS}}_{+}\rangle\!\rangle - \frac{a}{\sqrt{b}}|\tfrac{1}{10}{}^{\mathrm{NS}}_{+}\rangle\!\rangle - \frac{d}{\sqrt{c}}|\tfrac{7}{16}{}^{\mathrm{R}}_{+}\rangle\!\rangle + \frac{c}{\sqrt{d}}|\tfrac{3}{80}{}^{\mathrm{R}}_{+}\rangle\!\rangle \\ |\tfrac{\mathbf{1}}{\mathbf{10}}{}^{\widetilde{\mathrm{NS}}}\rangle &= \frac{b}{\sqrt{a}}|0^{\mathrm{NS}}_{+}\rangle\!\rangle - \frac{a}{\sqrt{b}}|\tfrac{1}{10}{}^{\mathrm{NS}}_{+}\rangle\!\rangle + \frac{d}{\sqrt{c}}|\tfrac{7}{16}{}^{\mathrm{R}}_{+}\rangle\!\rangle - \frac{c}{\sqrt{d}}|\tfrac{3}{80}{}^{\mathrm{R}}_{+}\rangle\!\rangle \\ |\tfrac{\mathbf{7}}{\mathbf{16}}{}^{\mathrm{R}}\rangle &= \frac{c}{\sqrt{a}}|0^{\mathrm{NS}}_{-}\rangle\!\rangle - \frac{d}{\sqrt{b}}|\tfrac{1}{10}{}^{\mathrm{NS}}_{-}\rangle\!\rangle \\ |\tfrac{\mathbf{3}}{\mathbf{80}}{}^{\mathrm{R}}\rangle &= \frac{d}{\sqrt{a}}|0^{\mathrm{NS}}_{-}\rangle\!\rangle + \frac{c}{\sqrt{b}}|\tfrac{1}{10}{}^{\mathrm{NS}}_{-}\rangle\!\rangle. \end{aligned} \tag{4.3}$$

Using (3.19), we obtain the g-factors

$$g_{\mathbf{0}^{\mathrm{NS}}} = g_{\mathbf{0}^{\widetilde{\mathrm{NS}}}} = \sqrt{a} \qquad g_{\frac{\mathbf{1}}{\mathbf{10}}^{\mathrm{NS}}} = g_{\frac{\mathbf{1}}{\mathbf{10}}^{\widetilde{\mathrm{NS}}}} = \frac{b}{\sqrt{a}} \qquad g_{\frac{\mathbf{7}}{\mathbf{16}}^{\mathrm{R}}} = \frac{c}{\sqrt{a}} \qquad g_{\frac{\mathbf{3}}{\mathbf{80}}^{\mathrm{R}}} = \frac{d}{\sqrt{a}}. \tag{4.4}$$

Let us compare these results with those [5, 14] obtained from the $\mathcal{M}(4/5)$ description. The $\mathcal{M}(4/5)$ Kac table is given in table 2.

Table 2. Kac table for $\mathcal{M}(4/5)$.

$\frac{3}{2}$	$\frac{3}{5}$	$\frac{1}{10}$	0
$\frac{7}{16}$	$\frac{3}{80}$	$\frac{3}{80}$	$\frac{7}{16}$
0	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{2}$

The modular S matrix is

$$S = \begin{pmatrix} a & b & b & a & c & d \\ b & -a & -a & b & -d & c \\ b & -a & -a & b & d & -c \\ a & b & b & a & -c & -d \\ c & -d & d & -c & 0 & 0 \\ d & c & -c & -d & 0 & 0 \end{pmatrix} \tag{4.5}$$

where the rows and columns are labelled by the highest weights $(0, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}, \frac{7}{16}, \frac{3}{80})$, and a–d are given by (4.2). As follows from (1.10), the Cardy states are given by[4]

$$\begin{aligned}
|\mathbf{0}\rangle &= \sqrt{a}|0\rangle\rangle + \sqrt{b}|\tfrac{1}{10}\rangle\rangle + \sqrt{b}|\tfrac{3}{5}\rangle\rangle + \sqrt{a}|\tfrac{3}{2}\rangle\rangle + \sqrt{c}|\tfrac{7}{16}\rangle\rangle + \sqrt{d}|\tfrac{3}{80}\rangle\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{2}}\rangle &= \sqrt{a}|0\rangle\rangle + \sqrt{b}|\tfrac{1}{10}\rangle\rangle + \sqrt{b}|\tfrac{3}{5}\rangle\rangle + \sqrt{a}|\tfrac{3}{2}\rangle\rangle - \sqrt{c}|\tfrac{7}{16}\rangle\rangle - \sqrt{d}|\tfrac{3}{80}\rangle\rangle \\
|\tfrac{\mathbf{1}}{\mathbf{10}}\rangle &= \frac{b}{\sqrt{a}}|0\rangle\rangle - \frac{a}{\sqrt{b}}|\tfrac{1}{10}\rangle\rangle - \frac{a}{\sqrt{b}}|\tfrac{3}{5}\rangle\rangle + \frac{b}{\sqrt{a}}|\tfrac{3}{2}\rangle\rangle - \frac{d}{\sqrt{c}}|\tfrac{7}{16}\rangle\rangle + \frac{c}{\sqrt{d}}|\tfrac{3}{80}\rangle\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{5}}\rangle &= \frac{b}{\sqrt{a}}|0\rangle\rangle - \frac{a}{\sqrt{b}}|\tfrac{1}{10}\rangle\rangle - \frac{a}{\sqrt{b}}|\tfrac{3}{5}\rangle\rangle + \frac{b}{\sqrt{a}}|\tfrac{3}{2}\rangle\rangle + \frac{d}{\sqrt{c}}|\tfrac{7}{16}\rangle\rangle - \frac{c}{\sqrt{d}}|\tfrac{3}{80}\rangle\rangle \\
|\tfrac{\mathbf{7}}{\mathbf{16}}\rangle &= \frac{c}{\sqrt{a}}|0\rangle\rangle - \frac{d}{\sqrt{b}}|\tfrac{1}{10}\rangle\rangle + \frac{d}{\sqrt{b}}|\tfrac{3}{5}\rangle\rangle - \frac{c}{\sqrt{a}}|\tfrac{3}{2}\rangle\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{80}}\rangle &= \frac{d}{\sqrt{a}}|0\rangle\rangle + \frac{c}{\sqrt{b}}|\tfrac{1}{10}\rangle\rangle - \frac{c}{\sqrt{b}}|\tfrac{3}{5}\rangle\rangle - \frac{d}{\sqrt{a}}|\tfrac{3}{2}\rangle\rangle.
\end{aligned} \tag{4.6}$$

We observe that the two modular S matrices (4.1) and (4.5) are related by a unitary transformation, due to the relation of the corresponding characters [21]

$$\begin{aligned}
&\chi_0^{\mathrm{NS}}(q) = \chi_0(q) + \chi_{\frac{3}{2}}(q) \qquad \chi_0^{\widetilde{\mathrm{NS}}}(q) = \chi_0(q) - \chi_{\frac{3}{2}}(q) \\
&\chi_{\frac{1}{10}}^{\mathrm{NS}}(q) = \chi_{\frac{1}{10}}(q) + \chi_{\frac{3}{5}}(q) \qquad \chi_{\frac{1}{10}}^{\widetilde{\mathrm{NS}}}(q) = \chi_{\frac{1}{10}}(q) - \chi_{\frac{3}{5}}(q) \\
&\chi_{\frac{7}{16}}^{\mathrm{R}}(q) = \chi_{\frac{7}{16}}(q) \qquad \chi_{\frac{3}{80}}^{\mathrm{R}}(q) = \chi_{\frac{3}{80}}(q).
\end{aligned} \tag{4.7}$$

Moreover, the Cardy states (4.3) and (4.6) can be seen to coincide, upon identifying the Ishibashi states

$$|0_{\pm}^{\mathrm{NS}}\rangle\rangle = |0\rangle\rangle \pm |\tfrac{3}{2}\rangle\rangle \qquad |\tfrac{1}{10}{}_{\pm}^{\mathrm{NS}}\rangle\rangle = |\tfrac{1}{10}\rangle\rangle \pm |\tfrac{3}{5}\rangle\rangle \qquad |\tfrac{7}{16}{}_{+}^{\mathrm{R}}\rangle\rangle = |\tfrac{7}{16}\rangle\rangle \qquad |\tfrac{3}{80}{}_{+}^{\mathrm{R}}\rangle\rangle = |\tfrac{3}{80}\rangle\rangle. \tag{4.8}$$

Evidently, whether we use the $\mathcal{M}(4/5)$ or $\mathcal{SM}(3/5)$ description, the Hamiltonian is the same, and so are the Cardy states and corresponding g factors. The two descriptions correspond to two equivalent bases.

5. Discussion

We have proposed (3.12) a supersymmetric generalization of Cardy's equation, and we have solved it for the consistent superconformal boundary states (3.15), (3.16), (3.18), thereby

[4] Our notation is related to Chim's [14] $C = \sqrt{\frac{\sin(\pi/5)}{\sqrt{5}}}$, $\eta = \sqrt{\frac{\sin(2\pi/5)}{\sin(\pi/5)}}$ by

$$a = C^2 \qquad b = C^2\eta^2 \qquad c = C^2\sqrt{2} \qquad d = C^2\eta^2\sqrt{2}.$$

classifying the possible superconformal BCs. In particular, there are $\widetilde{\mathrm{NS}}$ boundary states in addition to the NS and R states.

Having a better understanding of BCs in boundary superconformal field theories, one is in a better position to investigate integrable perturbations of these theories, and treat problems such as RG boundary flows.

For simplicity, we have restricted ourselves here to the unitary superconformal minimal models $\mathcal{SM}(p/p+2)$ with p odd. It should be possible to extend our analysis to the models with p even, and, in fact, to general (nonunitary) models $\mathcal{SM}(p/q)$. Also, a similar analysis should be possible for $N = 2$ superconformal models, which are important for superstring compactifications with spacetime supersymmetry [26, 27].

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Appendix. Ising model

Although the critical IM (i.e. the conformal minimal model $\mathcal{M}(3/4)$) does not have superconformal symmetry, it does have NS and R sectors. Here we work out explicitly how these sectors 'transform' between the open and closed channels of the cylinder. Because the IM is a free-field theory, the computations are particularly simple. Nevertheless, this exercise is useful, since it gives insight into how to treat the sectors of a superconformal model. Although the IM on a cylinder has already been studied extensively [5, 28–30], this particular aspect does not seem to have been emphasized before.

The critical two-dimensional IM is described by a free Majorana spinor field, whose two components we denote by $\psi(x, y)$ and $\bar{\psi}(x, y)$. We consider this model on the cylinder shown in figure 1, with $x \in [0, \mathsf{L}]$ the coordinate along the axis, and $y \in [0, \mathsf{R}]$ the coordinate along the circumference.

A.1. Open channel

In the open channel, we regard x as the space coordinate and y as the time coordinate. The time coordinate thus has period R, corresponding to the temperature $T = 1/\mathsf{R}$. The conformally invariant spatial BCs are [28]

$$\psi(0, y) + a\mathrm{i}\bar{\psi}(0, y) = 0 \qquad \psi(\mathsf{L}, y) - b\mathrm{i}\bar{\psi}(\mathsf{L}, y) = 0 \tag{A.1}$$

where $a, b = +1$ corresponds to 'fixed' BCs, and $a, b = -1$ corresponds to 'free' BCs.

Our first task is to find appropriate mode expansions for the fields ψ and $\bar{\psi}$. To this end, we recall that the overall relative sign between these fields is a matter of convention. We can therefore redefine $\bar{\psi}(x, y)$ such that

$$\psi(0, y) = \mathrm{i}\bar{\psi}(0, y) \tag{A.2}$$

which implies

$$\psi(\mathsf{L}, y) = -\mathrm{i}\frac{b}{a}\bar{\psi}(\mathsf{L}, y). \tag{A.3}$$

Proceeding as in the case of the superstring [6], we extend the definition of x to $[-\mathsf{L}, \mathsf{L}]$ and define the new field

$$\Psi(x, y) = \begin{cases} \psi(x, y) & \text{if} \quad x \in [0, \mathsf{L}] \\ \mathrm{i}\bar{\psi}(-x, y) & \text{if} \quad x \in [-\mathsf{L}, 0]. \end{cases} \tag{A.4}$$

This definition is consistent by virtue of equation (A.2). It follows that $\Psi(x, y)$ obeys the (quasi-) periodicity condition

$$\Psi(\mathsf{L}, y) = -\frac{b}{a}\Psi(-\mathsf{L}, y). \tag{A.5}$$

Thus, Ψ is periodic (R) if $a = -b$, and Ψ is antiperiodic (NS) if $a = b$. Note that a given set (a, b) of BCs is compatible with only one (R or NS) sector. The field Ψ has the standard mode expansion

$$\Psi(x, y) = \frac{1}{\sqrt{2\mathsf{L}}}\sum_k b_k \mathrm{e}^{-\mathrm{i}\frac{\pi}{\mathsf{L}}k(x+\mathrm{i}y)} \qquad \{b_k, b_l\} = \delta_{k+l,0} \tag{A.6}$$

with $k \in \mathsf{Z}$ for R, and $k \in \mathsf{Z} + \frac{1}{2}$ for NS. It follows that the sought-after mode expansions for ψ and $\bar{\psi}$ are

$$\begin{aligned} \psi(x, y) &= \frac{1}{\sqrt{2\mathsf{L}}}\sum_k b_k \mathrm{e}^{-\mathrm{i}\frac{\pi}{\mathsf{L}}k(x+\mathrm{i}y)} \\ \bar{\psi}(x, y) &= -\frac{\mathrm{i}}{\sqrt{2\mathsf{L}}}\sum_k b_k \mathrm{e}^{-\mathrm{i}\frac{\pi}{\mathsf{L}}k(-x+\mathrm{i}y)}. \end{aligned} \tag{A.7}$$

There is only one independent set of modes $\{b_k\}$ in the open channel.

The Hamiltonian H^{open} is

$$H^{\text{open}} = \frac{\pi}{\mathsf{L}}\left(e_0 + \sum_{k>0} k b_{-k} b_k\right) = \frac{\pi}{\mathsf{L}}\left(L_0 - \frac{c}{24}\right) \tag{A.8}$$

with

$$e_0^{\text{NS}} = -\tfrac{1}{48} \qquad e_0^{\text{R}} = \tfrac{1}{24} \tag{A.9}$$

and $c = \frac{1}{2}$. Standard computations give the partition functions

$$\begin{aligned} \mathrm{tr}_{\text{NS}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= q^{-\frac{1}{48}}\prod_{n=0}^{\infty}\left(1 + q^{\frac{1}{2}+n}\right) \\ \mathrm{tr}_{\text{NS}}(-1)^{\text{F}}\mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= q^{-\frac{1}{48}}\prod_{n=0}^{\infty}\left(1 - q^{\frac{1}{2}+n}\right) \\ \mathrm{tr}_{\text{R}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= 2q^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(1 + q^{n}\right) \\ \mathrm{tr}_{\text{R}}(-1)^{\text{F}}\mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= 0 \end{aligned} \tag{A.10}$$

where $q = \mathrm{e}^{-\pi\mathsf{R}/\mathsf{L}}$ and F is the fermion number operator. For the case of the IM, the Virasoro algebra has three irreducible representations with highest weights 0, 1/2, 1/16; the corresponding characters (1.5) are given by (see, e.g. [2])

$$\begin{aligned} \chi_0(q) &= \tfrac{1}{2}q^{-\frac{1}{48}}\left(\prod_{n=0}^{\infty}\left(1 + q^{\frac{1}{2}+n}\right) + \prod_{n=0}^{\infty}\left(1 - q^{\frac{1}{2}+n}\right)\right) \\ \chi_{\frac{1}{2}}(q) &= \tfrac{1}{2}q^{-\frac{1}{48}}\left(\prod_{n=0}^{\infty}\left(1 + q^{\frac{1}{2}+n}\right) - \prod_{n=0}^{\infty}\left(1 - q^{\frac{1}{2}+n}\right)\right) \\ \chi_{\frac{1}{16}}(q) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 + q^{n}). \end{aligned} \tag{A.11}$$

The partition functions therefore have the following expressions in terms of characters:

$$\begin{aligned} \mathrm{tr}_{\mathrm{NS}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= \chi_0(q) + \chi_{\frac{1}{2}}(q) \\ \mathrm{tr}_{\mathrm{NS}}(-1)^{\mathrm{F}} \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= \chi_0(q) - \chi_{\frac{1}{2}}(q) \\ \mathrm{tr}_{\mathrm{R}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= 2\chi_{\frac{1}{16}}(q). \end{aligned} \tag{A.12}$$

The modular transformation law (1.6) for the characters, together with the explicit modular S matrix for the case of the IM (see, e.g. [5]), imply

$$\begin{aligned} \chi_0(q) + \chi_{\frac{1}{2}}(q) &= \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}) \\ \chi_0(q) - \chi_{\frac{1}{2}}(q) &= \sqrt{2}\chi_{\frac{1}{16}}(\tilde{q}) \\ \chi_{\frac{1}{16}}(q) &= \frac{1}{\sqrt{2}}\big(\chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q})\big) \end{aligned} \tag{A.13}$$

where $\tilde{q} = \mathrm{e}^{-4\pi\mathsf{L}/\mathsf{R}}$. We conclude that the partition functions are given by

$$\begin{aligned} \mathrm{tr}_{\mathrm{NS}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}) \\ \mathrm{tr}_{\mathrm{NS}}(-1)^{\mathrm{F}} \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= \sqrt{2}\chi_{\frac{1}{16}}(\tilde{q}) \\ \mathrm{tr}_{\mathrm{R}}\, \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= \sqrt{2}\big(\chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q})\big) \\ \mathrm{tr}_{\mathrm{R}}(-1)^{\mathrm{F}} \mathrm{e}^{-\mathsf{R}H^{\text{open}}} &= 0. \end{aligned} \tag{A.14}$$

A.2. Closed channel

In the closed channel, we regard x as the time coordinate and y as the space coordinate. Since y is periodic, the fields ψ and $\bar{\psi}$ can be either periodic (R) or anti-periodic (NS). These fields have the standard mode expansions

$$\begin{aligned} \psi(x, y) &= \frac{1}{\sqrt{\mathsf{R}}} \sum_k a_k \mathrm{e}^{-\mathrm{i}\frac{2\pi}{\mathsf{R}}k(y-\mathrm{i}x)} \qquad \{a_k, a_l\} = \delta_{k+l,0} \\ \bar{\psi}(x, y) &= \frac{1}{\sqrt{\mathsf{R}}} \sum_k \bar{a}_k \mathrm{e}^{-\mathrm{i}\frac{2\pi}{\mathsf{R}}k(-y-\mathrm{i}x)} \qquad \{\bar{a}_k, \bar{a}_l\} = \delta_{k+l,0} \qquad \{a_k, \bar{a}_l\} = 0 \end{aligned} \tag{A.15}$$

with $k \in \mathsf{Z}$ for R, and $k \in \mathsf{Z} + \frac{1}{2}$ for NS. There are two independent sets of modes in the closed channel.

The Hamiltonian H^{closed} is

$$H^{\text{closed}} = \frac{2\pi}{\mathsf{R}}\left(2e_0 + \sum_{k>0} k\,(a_{-k}a_k + \bar{a}_{-k}\bar{a}_k)\right) = \frac{2\pi}{\mathsf{R}}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \tag{A.16}$$

where again $k \in \mathsf{Z}$ for R, $k \in \mathsf{Z} + \frac{1}{2}$ for NS and e_0 is given in (A.9).

The BCs (A.2), (A.3) now correspond to initial and final conditions on states. Expressing these conditions in terms of modes, we are led to define (up to normalization) the boundary kets $|B_\pm\rangle$ and the corresponding bras $\langle B_\pm|$ as the solutions of the constraints [8]

$$(a_k - \mathrm{i}\gamma\bar{a}_{-k})\,|B_\gamma\rangle = 0 \qquad \langle B_\gamma|\,(a_{-k} + \mathrm{i}\gamma\bar{a}_k) = 0 \tag{A.17}$$

where $\gamma = \pm 1$. The solutions in the NS sector are given by

$$|B_\gamma^{\mathrm{NS}}\rangle = \mathrm{e}^{\mathrm{i}\gamma\sum_{k=\frac{1}{2}}^{\infty} a_{-k}\bar{a}_{-k}}|0\rangle \qquad \langle B_\gamma^{\mathrm{NS}}| = \langle 0|\mathrm{e}^{-\mathrm{i}\gamma\sum_{k=\frac{1}{2}}^{\infty} \bar{a}_k a_k} \tag{A.18}$$

where the NS vacuum state $|0\rangle$ satisfies $a_{-k}|0\rangle = 0, \bar{a}_{-k}|0\rangle = 0$ for $k > 0$. These states have even fermion parity

$$(-1)^{\mathrm{F}}|B_\gamma^{\mathrm{NS}}\rangle = |B_\gamma^{\mathrm{NS}}\rangle \tag{A.19}$$

where F is now the total fermion number operator in the NS sector,

$$F = \sum_{k=\frac{1}{2}}^{\infty} (a_{-k}a_k + \bar{a}_{-k}\bar{a}_k) . \tag{A.20}$$

The solutions of (A.17) in the R sector are given by

$$|B^{\mathrm{R}}_\gamma\rangle = \mathrm{e}^{\mathrm{i}\gamma \sum_{k=1}^{\infty} a_{-k}\bar{a}_{-k}} |\gamma\rangle \qquad \langle B^{\mathrm{R}}_\gamma| = \langle\gamma|\mathrm{e}^{-\mathrm{i}\gamma \sum_{k=1}^{\infty} \bar{a}_k a_k} \tag{A.21}$$

where the degenerate R vacuum states $|\pm\rangle$ satisfy $a_{-k}|\pm\rangle = 0$, $\bar{a}_{-k}|\pm\rangle = 0$ for $k > 0$, as well as

$$(a_0 - \mathrm{i}\gamma\bar{a}_0)\,|\gamma\rangle = 0. \tag{A.22}$$

An explicit representation for the zero modes is (see e.g. [2, 30])

$$a_0|\pm\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}}|\mp\rangle \qquad \bar{a}_0|\pm\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{4}}|\mp\rangle \tag{A.23}$$

using which one can readily verify (A.22). Moreover, $(2\mathrm{i}a_0\bar{a}_0)\,|\pm\rangle = \pm|\pm\rangle$. The total fermion parity operator $(-1)^{\mathrm{F}}$ in the R sector is given by

$$(-1)^{\mathrm{F}} = (2\mathrm{i}a_0\bar{a}_0)\,\mathrm{e}^{\mathrm{i}\pi \sum_{k=1}^{\infty}(a_{-k}a_k+\bar{a}_{-k}\bar{a}_k)} \tag{A.24}$$

(we choose the sign so that $|+\rangle$ has $(-1)^{\mathrm{F}} = 1$), and thus, the boundary states satisfy

$$(-1)^{\mathrm{F}}|B^{\mathrm{R}}_\pm\rangle = \pm|B^{\mathrm{R}}_\pm\rangle. \tag{A.25}$$

Using standard techniques, we find

$$\begin{aligned}
\langle B^{\mathrm{NS}}_\pm|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{NS}}_\pm\rangle &= \tilde{q}^{-\frac{1}{48}}\prod_{n=0}^{\infty}\left(1+\tilde{q}^{\frac{1}{2}+n}\right) = \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q})\\
\langle B^{\mathrm{NS}}_\mp|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{NS}}_\pm\rangle &= \tilde{q}^{-\frac{1}{48}}\prod_{n=0}^{\infty}\left(1-\tilde{q}^{\frac{1}{2}+n}\right) = \chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q})\\
\langle B^{\mathrm{R}}_\pm|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{R}}_\pm\rangle &= \tilde{q}^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+\tilde{q}^{n}\right) = \chi_{\frac{1}{16}}(\tilde{q})\\
\langle B^{\mathrm{R}}_\mp|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{R}}_\pm\rangle &= 0.
\end{aligned} \tag{A.26}$$

Recalling the results (A.14) from the open channel, we obtain the sought-after relations

$$\begin{aligned}
\mathrm{tr}_{\mathrm{NS}}\,\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}} &= \langle B^{\mathrm{NS}}_\pm|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{NS}}_\pm\rangle\\
\mathrm{tr}_{\mathrm{NS}}(-1)^{\mathrm{F}}\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}} &= \sqrt{2}\langle B^{\mathrm{R}}_\pm|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{R}}_\pm\rangle\\
\mathrm{tr}_{\mathrm{R}}\,\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}} &= \sqrt{2}\langle B^{\mathrm{NS}}_\mp|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{NS}}_\pm\rangle\\
\mathrm{tr}_{\mathrm{R}}(-1)^{\mathrm{F}}\mathrm{e}^{-\mathsf{R}H^{\mathrm{open}}} &= 0 = \langle B^{\mathrm{R}}_\mp|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|B^{\mathrm{R}}_\pm\rangle
\end{aligned} \tag{A.27}$$

which show explicitly how the NS and R sectors 'transform' between the open and closed channels of the cylinder[5]. Similar results are known in string theory.

We conclude this subsection by noting that the boundary states (A.18), (A.21) are closely related to the Ishibashi states (1.2). Namely,

$$|0\rangle\!\rangle = \tfrac{1}{2}\left(|B^{\mathrm{NS}}_+\rangle + |B^{\mathrm{NS}}_-\rangle\right) \qquad |\tfrac{1}{2}\rangle\!\rangle = \tfrac{1}{2}\left(|B^{\mathrm{NS}}_+\rangle - |B^{\mathrm{NS}}_-\rangle\right) \qquad |\tfrac{1}{16}\rangle\!\rangle = |B^{\mathrm{R}}_+\rangle. \tag{A.28}$$

Indeed, recalling that the boundary states satisfy (A.17) and that

$$L_n = \frac{1}{2}\sum_k \left(k + \frac{n}{2}\right) : a_{-k}a_{n+k} : \qquad \bar{L}_n = \frac{1}{2}\sum_k \left(k + \frac{n}{2}\right) : \bar{a}_{-k}\bar{a}_{n+k} : \tag{A.29}$$

[5] Numerical factors appear in these relations because the NS and R sectors are not irreducible representations of the Virasoro algebra and also the states $|B^{\mathrm{NS}}_\pm\rangle$ are not properly normalized (see equations (A.28) and (A.30) below).

one can easily show that the boundary states satisfy the constraint (1.1). Moreover, expanding the exponentials in the expressions (A.18), (A.21) and comparing the leading terms with (1.2), one can infer (A.28). Regarding the Ishibashi states as orthonormal vectors $(i, j) = \delta_{ij}$, it follows from (A.28) that the boundary states have the normalization

$$\left(B_\pm^{\mathrm{NS}}, B_\pm^{\mathrm{NS}}\right) = 2 \qquad \left(B_\pm^{\mathrm{R}}, B_\pm^{\mathrm{R}}\right) = 1. \tag{A.30}$$

Strictly speaking, the Ishibashi states and boundary states $|B_\pm\rangle$ are *not* normalizable. However, one can define an inner product [5, 10] and argue

$$\frac{\left(B_\pm^{\mathrm{NS}}, B_\pm^{\mathrm{NS}}\right)}{(0,0)} = \lim_{q\to1} \frac{\langle B_\pm^{\mathrm{NS}}|q^{L_0+\bar{L}_0}|B_\pm^{\mathrm{NS}}\rangle}{\langle\langle 0|q^{L_0+\bar{L}_0}|0\rangle\rangle} = \lim_{q\to1} \frac{\chi_0(q^2) + \chi_{\frac{1}{2}}(q^2)}{\chi_0(q^2)} = 2. \tag{A.31}$$

A.3. Consistent boundary states

Finally, it is also instructive to rederive Cardy's results for the consistent IM boundary states in our basis $|B_\pm\rangle$. We begin by rewriting the fundamental consistency constraint (1.3) as

$$\begin{aligned} &\mathrm{tr}_{\mathrm{NS}}\, \tfrac{1}{2}(1+(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} + \mathrm{tr}_{\mathrm{NS}}\, \tfrac{1}{2}(1-(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} \\ &\qquad + \mathrm{tr}_{\mathrm{R}}\, \tfrac{1}{2}(1+(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} = \langle\alpha|\mathrm{e}^{-\mathsf{L}H^{\mathrm{closed}}}|\beta\rangle. \end{aligned} \tag{A.32}$$

From the results (A.12), it is evident that

$$\begin{aligned} \mathrm{tr}_{\mathrm{NS}}\, \tfrac{1}{2}(1+(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} &= N_{\alpha\beta}^{0}\chi_0(q) \\ \mathrm{tr}_{\mathrm{NS}}\, \tfrac{1}{2}(1-(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} &= N_{\alpha\beta}^{\frac{1}{2}}\chi_{\frac{1}{2}}(q) \\ \mathrm{tr}_{\mathrm{R}}\, \tfrac{1}{2}(1+(-1)^{\mathrm{F}})\mathrm{e}^{-\mathsf{R}H_{\alpha\beta}^{\mathrm{open}}} &= N_{\alpha\beta}^{\frac{1}{16}}\chi_{\frac{1}{16}}(q) \end{aligned} \tag{A.33}$$

and so the LHS of (A.32) is indeed equal to $\sum_i N_{\alpha\beta}^i\chi_i(q)$. In the RHS of (A.32), we expand the boundary states in the basis $|B_\pm\rangle$ using

$$|\alpha\rangle = \tfrac{1}{2}\left(|B_+^{\mathrm{NS}}\rangle\langle B_+^{\mathrm{NS}}|\alpha\rangle + |B_-^{\mathrm{NS}}\rangle\langle B_-^{\mathrm{NS}}|\alpha\rangle\right) + |B_+^{\mathrm{R}}\rangle\langle B_+^{\mathrm{R}}|\alpha\rangle \tag{A.34}$$

keeping in mind the normalization (A.30). Then, making use also of the relations (A.27), we arrive at the Cardy equations

$$\begin{aligned} N_{\alpha\beta}^{0} &= \frac{1}{4}\langle\alpha|B_+^{\mathrm{NS}}\rangle\langle B_+^{\mathrm{NS}}|\beta\rangle + \frac{1}{4}\langle\alpha|B_-^{\mathrm{NS}}\rangle\langle B_-^{\mathrm{NS}}|\beta\rangle + \frac{1}{\sqrt{2}}\langle\alpha|B_+^{\mathrm{R}}\rangle\langle B_+^{\mathrm{R}}|\beta\rangle \\ N_{\alpha\beta}^{\frac{1}{2}} &= \frac{1}{4}\langle\alpha|B_+^{\mathrm{NS}}\rangle\langle B_+^{\mathrm{NS}}|\beta\rangle + \frac{1}{4}\langle\alpha|B_-^{\mathrm{NS}}\rangle\langle B_-^{\mathrm{NS}}|\beta\rangle - \frac{1}{\sqrt{2}}\langle\alpha|B_+^{\mathrm{R}}\rangle\langle B_+^{\mathrm{R}}|\beta\rangle \\ N_{\alpha\beta}^{\frac{1}{16}} &= \frac{1}{2\sqrt{2}}\langle\alpha|B_+^{\mathrm{NS}}\rangle\langle B_-^{\mathrm{NS}}|\beta\rangle + \frac{1}{2\sqrt{2}}\langle\alpha|B_-^{\mathrm{NS}}\rangle\langle B_+^{\mathrm{NS}}|\beta\rangle. \end{aligned} \tag{A.35}$$

Following Cardy [5], we define the states $|\boldsymbol{k}\rangle$ by $N_{\mathbf{0}\boldsymbol{k}}^i = \delta_k^i$, and we obtain

$$\begin{aligned} |\mathbf{0}\rangle &= \frac{1}{\sqrt{2}}|B_-^{\mathrm{NS}}\rangle + \frac{1}{\sqrt[4]{2}}|B_+^{\mathrm{R}}\rangle \\ |\tfrac{\mathbf{1}}{\mathbf{2}}\rangle &= \frac{1}{\sqrt{2}}|B_-^{\mathrm{NS}}\rangle - \frac{1}{\sqrt[4]{2}}|B_+^{\mathrm{R}}\rangle \\ |\tfrac{\mathbf{1}}{\mathbf{16}}\rangle &= |B_+^{\mathrm{NS}}\rangle. \end{aligned} \tag{A.36}$$

These states correspond to the BCs 'fixed +', 'fixed −' and 'free', respectively. The g-factor [12] of a boundary state $|\alpha\rangle$ is given by

$$g_\alpha = \langle\langle 0|\alpha\rangle = \tfrac{1}{2}\left(\langle B_+^{\mathrm{NS}}| + \langle B_-^{\mathrm{NS}}|\right)|\alpha\rangle. \tag{A.37}$$

We therefore obtain (again remembering the normalization (A.30)) the well known results

$$g_{\mathbf{0}} = g_{\frac{\mathbf{1}}{\mathbf{2}}} = \frac{1}{\sqrt{2}} \qquad g_{\frac{\mathbf{1}}{\mathbf{16}}} = 1. \tag{A.38}$$

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